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The statistical mechanics of the classical two-dimensional Coulomb gas is exactly solved*

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Abstract
The model under consideration is a classical 2D Coulomb gas of pointlike positive and negative unit charges, interacting via a logarithmic potential. In the whole stability range of temperatures, the equilibrium statistical mechanics of this fluid model is exactly solvable via an equivalence with the integrable 2D sine-Gordon field theory. The exact solution includes the bulk thermodynamics, special cases of the surface thermodynamics and the large-distance asymptotic behaviour of the two-body correlation functions.

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1. Introduction

The classical (i.e. non-quantum) equilibrium statistical mechanics deals in general with two basic kinds of models: discrete lattice systems and continuous fluids. In one spatial dimension (1D), both kinds of statistical models, considered with short-range as well as long-range pairwise interactions among constituents, are solvable in many cases [1]. In 2D, there is a large family of integrable lattice systems exactly solvable via the Bethe-ansatz method (see [2, 3]). On the other hand, there was no exactly solved fluid in more than 1D. The only partial exceptions were represented by 2D logarithmic models of Coulomb fluids, the one-component plasma [4] and the symmetric two-component plasma (or Coulomb gas) [5] in the point-particle limit, solvable at one special value of the dimensionless inverse temperature $\beta = 2$.

The situation has changed in very recent years. In a series of works, the bulk and surface thermodynamics as well as the large-distance asymptotic behaviour of the two-body correlation functions were derived exactly for the 2D Coulomb gas of pointlike particles, in

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the whole stability range of inverse temperatures $\beta < 2$. These results were obtained by mapping the Coulomb gas onto the 2D sine-Gordon theory with a conformal normalization of the cos-field, and subsequently applying techniques and recent achievements in that integrable theory.

The aim of this paper is to present the exact solution of the 2D Coulomb gas to the fluid community, in the language of fluid physics and in a way accessible to non-specialists in field theory. Unsolved problems and further potential developments are pointed out.

In section 2, we introduce the 2D Coulomb gas. Section 3 deals with its complete bulk thermodynamics, obtained from the mapping onto the bulk sine-Gordon field theory. Surface thermodynamic properties of a semi-infinite 2D Coulomb gas in contact with an impermeable (ideal dielectric or ideal conductor) wall, and the corresponding mapping onto integrable boundary sine-Gordon models are given in section 4. The large-distance asymptotic behaviour of two-body correlation functions in the 2D Coulomb gas is presented in section 5. Section 6 is devoted to miscellaneous topics and perspectives.

2. Basic facts about the 2D Coulomb gas

The symmetric Coulomb gas, defined in an infinite 2D space of points $\mathbf{r} \in R^2$, consists of point particles $\{i\}$ of charge $\{q_i = \pm 1\}$ immersed in a homogeneous medium of dielectric constant $= 1$. The interaction energy of particles is $\sum_{i<j} q_i q_j v(|\mathbf{r}_i - \mathbf{r}_j|)$, where the Coulomb potential v is the solution of the 2D Poisson equation

$$\Delta v(\mathbf{r}) = -2\pi\delta(\mathbf{r}). \tag{1}$$

Explicitly, $v(\mathbf{r}) = -\ln(|\mathbf{r}|/L)$ where L is a length scale. This definition of the Coulomb potential in 2D maintains many generic properties (e.g., sum rules) of 'real' 3D Coulomb fluids with the interaction potential $v(\mathbf{r}) = 1/|\mathbf{r}|$, $\mathbf{r} \in R^3$.

The model is treated as the classical one in thermodynamic equilibrium, via the grand canonical ensemble characterized by the (dimensionless) inverse temperature β and by the couple of particle fugacities $z_+(\mathbf{r}) = z_-(\mathbf{r}) = z$. We set the free length scale L to unity for simplicity; the true dimension of the rescaled z is then $[\text{length}]^{\beta/2-2}$. The grand partition function is defined by

$$\Xi = \sum_{N_+,N_-=0}^{\infty} \frac{1}{N_+!N_-!} \int \prod_{i=1}^{N} \left[\mathrm{d}^2 r_i\, z_{q_i}(\mathbf{r}_i)\right] \exp\left[-\beta \sum_{i<j} q_i q_j v(|\mathbf{r}_i - \mathbf{r}_j|)\right] \tag{2}$$

where N_+ (N_-) is the number of positive (negative) particles and $N = N_+ + N_-$. Many-particle densities are generated from Ξ in a standard way as functional derivatives with respect to the fugacity field $z_q(\mathbf{r})$, taken at the constant $z_+(\mathbf{r}) = z_-(\mathbf{r}) = z$. At the one-particle level, one introduces the number density of particles of one sign $n_q = \left\langle \sum_i \delta_{q,q_i}\delta(\mathbf{r} - \mathbf{r}_i)\right\rangle$. Due to the charge symmetry, $n_+ = n_- = n/2$ (n is the total density of particles). At the two-particle level, one introduces the two-body densities $n_{qq'}(|\mathbf{r}-\mathbf{r}'|) = \left\langle \sum_{i\neq j} \delta_{q,q_i}\delta(\mathbf{r}-\mathbf{r}_i)\delta_{q',q_j}\delta(\mathbf{r}'-\mathbf{r}_j)\right\rangle$. It is useful to consider also the pair distribution functions $g_{qq'}(r) = n_{qq'}(r)/(n_q n_{q'})$, the (truncated) correlation functions $h_{qq'}(r) = g_{qq'}(r) - 1$ and the Ursell functions $U_{qq'}(r) = n_q n_{q'} h_{qq'}(r)$.

The underlying system of pointlike particles is stable against the collapse of positive–negative pairs of charges provided that the corresponding Boltzmann factor $r^{-\beta}$ is integrable at short distances in 2D, i.e. for $\beta < 2$. To cross the collapse point $\beta = 2$, the pure Coulomb interaction has to be regularized by a short-distance repulsion, e.g., a hard-core potential of diameter σ around each particle (the particular choice of the short-distance regularization influences the results quantitatively, but not qualitatively). For small values of the dimensionless density $n\sigma^2$, the system remains in its conducting phase (an external charge is perfectly screened by the system charges) up to the Kosterlitz–Thouless (KT) transition of infinite order [6] at a specific density-dependent β_{KT}; $\beta_{\mathrm{KT}} = 4$ in the low-density limit. In the insulating phase $\beta > \beta_{\mathrm{KT}}$, the system charges form dipoles and no longer screen an external charge. At high enough density, the KT critical line splits into a first-order liquid–gas coexistence curve [7]. In what follows, we shall restrict ourselves to the point-particle Coulomb gas in the stability region $\beta < 2$.

A complete exact analysis can be done in two cases: in the high-temperature Debye–Hückel limit $\beta \to 0$, and just at the collapse point $\beta = 2$ [5] which corresponds to the free-fermion point of an equivalent 2D Thirring model. Although, at a given z, the free energy diverges, Ursell functions are finite at $\beta = 2$.

As concerns exact information valid in the whole stability region $\beta < 2$, through a simple scaling argument, the exact equation of state for the pressure P, $\beta P = n(1 - \beta/4)$, has been known for a long time [8]. While the density derivatives of the Helmholtz free energy, such as the pressure, can all be calculated exactly, the temperature derivatives, such as the internal energy or the specific heat, are nontrivial quantities. Their evaluation can be based on an explicit density–fugacity (n, z) relationship. The latter was constructed systematically around the $\beta \to 0$ point by using a bond-renormalized Mayer expansion in density [9]: the original bonds $-\beta v(r)$ are summed up in series to produce the renormalized bonds of strength $-\beta K_0(\kappa r)$, where K_0 denotes a modified Bessel function and $\kappa = (2\pi\beta n)^{1/2}$ is the inverse Debye length. The cluster integrals converge in the renormalized format. The first few integrals imply

$$\frac{n^{1-\beta/4}}{z} = 2\beta^{\beta/4} \exp\left\{\left[2C + \ln\left(\frac{\pi}{2}\right)\right]\frac{\beta}{4} + \frac{7}{6}\zeta(3)\left(\frac{\beta}{4}\right)^3 + \zeta(3)\left(\frac{\beta}{4}\right)^4 + O(\beta^5)\right\} \tag{3}$$

where C is the Euler number and ζ denotes the Riemann zeta function.

3. Bulk thermodynamics

The 2D Coulomb gas is equivalent to the 2D sine-Gordon model [10]. Introducing the microscopic charge density $\hat{\rho}(\mathbf{r}) = \sum_{i=1}^{N} q_i \delta(\mathbf{r} - \mathbf{r}_i)$, the interaction energy can be written as

$$E = \frac{1}{2}\int \mathrm{d}^2r\, \mathrm{d}^2r'\, \hat{\rho}(\mathbf{r}) v(|\mathbf{r} - \mathbf{r}'|)\hat{\rho}(\mathbf{r}') - \frac{1}{2}Nv(0). \tag{4}$$

Let us forget for a while that $v(0)$ diverges, and renormalize the fugacity by the self-energy term $\exp[-\beta v(0)/2]$, without changing the z-notation. Using the fact that $-\Delta/(2\pi)$ is the inverse operator of $v(\mathbf{r})$ (see equation (1)), the grand partition function (2) with $z_q(\mathbf{r}) = z$ can be turned via the Hubbard–Stratonovich transformation into

$$\Xi(z) = \frac{\int \mathcal{D}\phi \exp(-S(z))}{\int \mathcal{D}\phi \exp(-S(0))} \tag{5a}$$

where

$$S(z) = \int \mathrm{d}^2r \left[\frac{1}{16\pi}(\nabla\phi)^2 - 2z\cos(b\phi)\right] \qquad b = \left(\frac{\beta}{4}\right)^{1/2} \tag{5b}$$

is the Euclidean action of the 2D sine-Gordon theory. Here, $\phi(\mathbf{r})$ is a real scalar field and $\int \mathcal{D}\phi$ denotes the functional integration over this field. The many-particle densities are expressible as averages over the sine-Gordon action as follows:

$$n_q = z_q\langle \mathrm{e}^{\mathrm{i}qb\phi}\rangle \qquad n_{qq'}(|\mathbf{r} - \mathbf{r}'|) = z_q z_{q'}\langle \mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\,\mathrm{e}^{\mathrm{i}q'b\phi(\mathbf{r}')}\rangle \tag{6}$$

etc. The parameter z in (5b), i.e. the fugacity renormalized by the diverging self-energy term, gets a precise meaning when one fixes the normalization of the coupled cos-field. In the Coulomb system, the behaviour of the two-body density for oppositely charged particles is dominated at short distances by the Boltzmann factor of the Coulomb potential, $n_{+-}(\mathbf{r}, \mathbf{r}') \sim z_+ z_- |\mathbf{r} - \mathbf{r}'|^{-\beta}$ as $|\mathbf{r} - \mathbf{r}'| \to 0$. With regard to (6), the mapping is supplemented by the short-distance normalization

$$\langle \mathrm{e}^{\mathrm{i}b\phi(\mathbf{r})} \, \mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r})} \rangle \sim |\mathbf{r} - \mathbf{r}'|^{-4b^2} \qquad \text{as} \quad |\mathbf{r} - \mathbf{r}'| \to 0 \tag{7}$$

which was usually omitted in the statmech literature. Under this short-distance normalization, the divergent self-energy factor disappears from statistical relations calculated within the sine-Gordon representation. This can be easily verified in the Debye–Hückel limit $\beta \to 0$, when $\cos(b\phi) \sim 1 - b^2\phi^2/2$, and the consequent Gaussian field theory reproduces the n, z relation (3) up to the linear β-term in the exponential.

In the classical limit of the sine-Gordon theory, only such configurations of the ϕ-field are considered which fulfil the equation of motion $\delta S = 0$. This classical limit is integrable [11], i.e. there exists an infinite sequence of conserved quantities. Due to the discrete symmetry $\phi \to \phi + 2\pi n/b$ (n integer), the model has an infinite number of vacua at $\phi_n = 2\pi n/b$. The basic 'particles', the soliton S and antisoliton $\bar{S}$ pair of equal masses M, interpolate between two neighbouring vacua. The S–$\bar{S}$ pair can create bound states, called breather particles $\{B\}$. The sine-Gordon model is integrable at the full 'quantum level' (all configurations of the ϕ-field are considered) as well [11], with the same particle spectrum. The essential difference between the classical and quantum theories is that the breathers become quantized, $\{B_j;\ j = 1, 2, \ldots < 1/\xi\}$, and their number depends on the inverse of the parameter $\xi = b^2/(1 - b^2)$. The mass of the B_j-breather is given by

$$m_j = 2M \sin\left(\frac{\pi\xi}{2} j\right) \tag{8}$$

and this breather disappears from the spectrum just when $m_j = 2M$. Note that breathers exist only in the stability region of the Coulomb gas $0 < b^2 < 1/2$ $(0 < \beta < 2)$. The lightest B_1-breather disappears just at $b^2 = 1/2$ $(\beta = 2)$, which is the field-theoretical manifestation of the collapse phenomenon.

Using the thermodynamic Bethe ansatz, the dimensionless specific grand potential

$$\lim_{V\to\infty} \frac{1}{V} \ln \Xi = \frac{m_1^2}{8\sin(\pi\xi)} \tag{9}$$

was found by Destri and de Vega [12]. Under the conformal normalization (7), the relationship between the soliton mass M and the fugacity z was established in [13],

$$z = \frac{\Gamma(b^2)}{\pi\Gamma(1 - b^2)} \left[M \frac{\sqrt{\pi}\,\Gamma((1+\xi)/2)}{2\Gamma(\xi/2)} \right]^{2-2b^2} \tag{10}$$

where Γ stands for the Gamma function. Equations (8)–(10) constitute a complete set to be solved for the exact n, z relationship [9]:

$$\frac{n^{1-\beta/4}}{z} = 2\left(\frac{\pi\beta}{8}\right)^{\beta/4} \frac{\Gamma(1-\beta/4)}{\Gamma(1+\beta/4)} \left[F\left(\frac{1}{2}, \frac{\beta}{4-\beta}; 1 + \frac{\beta}{2(4-\beta)}; 1\right) \right]^{1-\beta/4} \tag{11}$$

where $F \equiv {}_2F_1$ is the hypergeometric function. The expansion of the rhs around $\beta \to 0$ reproduces correctly the first terms of the renormalized expansion (3). For fixed z, the particle density given by (11) exhibits the expected collapse singularity $n \sim 4\pi z^2/(2 - \beta)$ as $\beta \to 2^-$. This behaviour can be derived independently by combining an electroneutrality sum rule, $-qn_q = \sum_{q'=\pm} q' \int \mathrm{d}^2 r \, n_{qq'}(r)$, with the short-distance asymptotic behaviour of

$n_{+-}(r)$ discussed above. Since the derivation of formula (10) was based on special analyticity assumptions, the check of the results from both sides of the stability interval is important. Without noticing it, such checks are made for all presented results.

Based on the explicit n, z relation, one can pass by using the Legendre transformation from the grand canonical to the canonical ensemble, to obtain the Helmholtz free energy. We present the explicit result for the excess specific heat at constant volume per particle [9],

$$\frac{c_V^{\text{ex}}}{k_B} = \frac{\beta}{4} + \frac{4}{4-\beta} + \frac{\beta^2}{16}\left[\psi'\left(1-\frac{\beta}{4}\right) - \psi'\left(1+\frac{\beta}{4}\right)\right] - \frac{2\beta^2}{(4-\beta)^3}\left[\psi'\left(\frac{2}{4-\beta}\right)\right.$$
$$\left. - \psi'\left(\frac{8-\beta}{8-2\beta}\right)\right] - \frac{4\pi^2\beta^2}{(4-\beta)^3}\frac{\cos(\pi\beta/(4-\beta))}{\sin^2(\pi\beta/(4-\beta))}. \tag{12}$$

Here, $\psi(x) = \mathrm{d}[\ln\Gamma(x)]/\mathrm{d}x$ is the psi function and $\psi'(x) = \sum_{i=0}^{\infty} 1/(x+i)^2$. As $\beta \to 2^-$, the c_V^{ex}/k_B exhibits the expected singularity of type $2/(2-\beta)^2$. Note that the specific heat is independent of the particle density, which is a peculiarity of the 2D Coulomb gases.

4. Surface thermodynamics

Let us now consider a semi-infinite 2D Coulomb gas in the Cartesian half-space $x > 0$, in contact with a hard wall of dielectric constant ϵ_W in the complementary half-space $x < 0$. The presence of the dielectric wall manifests itself through particle images [14]: the particle of charge q at position $\mathbf{r} = (x, y)$ has the image of charge q^* (dependent on ϵ_W) at $\mathbf{r}^* = (-x, y)$. We will consider two particular cases: the ideal dielectric wall ($\epsilon_W = 0$) with image charges $q^* = q$ and the ideal conductor wall ($\epsilon_W \to \infty$) with image charges $q^* = -q$. Let us introduce the microscopic charge plus image-charge density $\hat{\rho}(\mathbf{r}) = \sum_{i=1}^{N} q_i\,[\delta(x-x_i) \pm \delta(x+x_i)]\,\delta(y-y_i)$; hereinafter, the upper (+) sign corresponds to $\epsilon_W = 0$ and the lower (−) sign to $\epsilon_W \to \infty$. The interaction energy of the particle–image system can be written in both cases as

$$E = \frac{1}{4}\int \mathrm{d}^2 r \int \mathrm{d}^2 r'\, \hat{\rho}(\mathbf{r}) v(|\mathbf{r}-\mathbf{r}'|)\hat{\rho}(\mathbf{r}') - \frac{1}{2}Nv(0) \tag{13}$$

where the integrations over $\mathbf{r}$ and $\mathbf{r}'$ are taken over the whole 2D space.

The form of the interaction energy (13) resembles that in (4), and one can proceed in close analogy with the bulk mapping. The grand partition function is expressible as

$$\Xi(z) = \frac{\int \mathcal{D}\phi \exp(-S(z))}{\int \mathcal{D}\phi \exp(-S(0))} \tag{14a}$$

where the ϕ-field is defined in the whole 2D space and the nonlocal action reads

$$S(z) = \int \mathrm{d}^2 r \left[\frac{1}{16\pi}(\nabla\phi)^2 - 2z\cos\left(\frac{b}{\sqrt{2}}[\phi(x,y) \pm \phi(-x,y)]\right)\right] \tag{14b}$$

$b = \sqrt{\beta/4}$. To make this field theory local, we introduce two new fields

$$\phi_e(x,y) = \tfrac{1}{\sqrt{2}}[\phi(x,y) + \phi(-x,y)] \qquad \phi_o(x,y) = \tfrac{1}{\sqrt{2}}[\phi(x,y) - \phi(-x,y)] \tag{15}$$

defined only in the positive $x \geqslant 0$ half-space. The 'even' field has a Neumann boundary condition, $\partial_x\phi_e|_{x=0} = 0$, and the 'odd' field a Dirichlet boundary condition, $\phi_o|_{x=0} = 0$. Since $\int \mathrm{d}^2 r (\nabla\phi)^2 = \int_{x>0} \mathrm{d}^2 r[(\nabla\phi_e)^2 + (\nabla\phi_o)^2]$, the fields ϕ_e and ϕ_o are decoupled in the action (14b). The field, contributing only by its free-field part, disappears from (14a)

by numerator–denominator cancellation. Consequently, renaming the kept field as ϕ, we arrive at

$$\Xi(z) = \frac{\int \mathcal{D}\phi \exp(-S(z))}{\int \mathcal{D}\phi \exp(-S(0))} \qquad S(z) = \int_{x>0} \mathrm{d}^2 r \left[\frac{1}{16\pi} (\nabla\phi)^2 - 2z\cos(b\phi) \right]. \tag{16}$$

Here, the ϕ-field has the boundary condition: $\phi|_{x=0} = 0$ for the ideal conductor wall; $\partial_x \phi|_{x=0} = 0$ for the ideal dielectric wall. The mapping onto the boundary sine-Gordon model is supplemented by the short-distance normalization (7). Both cases under consideration belong to the integrable boundary field theories [15]. The thermodynamic quantity of interest is the surface tension γ, which characterizes the surface part of the grand potential $\Omega = -\beta^{-1} \ln \Xi$.

The problem of the ideal conductor wall was solved via a lattice regularization of the boundary sine-Gordon model [16], namely the XXZ Heisenberg quantum chain in boundary magnetic fields. The surface tension was obtained in terms of the soliton mass M as follows:

$$\beta\gamma_{\mathrm{cond}} = \frac{M}{4} \left\{ 1 - \tan\left(\frac{\pi\beta}{2(4-\beta)} \right) - \left[\cos\left(\frac{\pi\beta}{2(4-\beta)} \right) \right]^{-1} \right\}. \tag{17}$$

The surface collapse is governed by the interaction Boltzmann factor of a particle with its self-image, $x^{-\beta/2}$. The 1D integral $\int \mathrm{d}x\, x^{-\beta/2}$ diverges at short distances at point $\beta = 2$ (which coincides with the bulk collapse point), and this is indeed the radius of convergence of γ_{cond}.

The problem of the ideal dielectric wall was solved by exploring a 'reflection' relationship between the Liouville and sine-Gordon theories [17]. The result is

$$\beta\gamma_{\mathrm{diel}} = \frac{M}{4} \left\{ 1 + \tan\left(\frac{\pi\beta}{2(4-\beta)} \right) - \left[\cos\left(\frac{\pi\beta}{2(4-\beta)} \right) \right]^{-1} \right\}. \tag{18}$$

At $\beta = 2$, γ_{diel} keeps a finite value [18]. The analytic continuation of (18) beyond the bulk collapse point predicts a surface collapse at $\beta = 3$. Such a phenomenon is due to the paradoxical short-distance attraction of a particle with its own image charge of the same sign [17].

The surface thermodynamics of a plain hard wall ($\epsilon_W = 1$), which is the last exactly solvable case at the free-fermion point $\beta = 2$ [19], is an open problem.

5. Large-distance behaviour of particle correlations

In a 2D integrable theory characterized by a 'particle' spectrum, correlation functions of local fields can be written as an infinite convergent series over multi-particle intermediate states, in terms of the corresponding form factors. The form-factor representation is especially useful at large distances, since the dominant contribution of the series comes from an intermediate state with the minimum value of the total particle mass. In the sine-Gordon theory, for topological reasons, solitons S and antisolitons $\bar{S}$ coexist in pairs, the total mass of the pair being $2M$. The breathers $\{B_j\}$, when they exist, have lighter masses (see formula (8)). The lightest B_1-breather with mass $m_1 = 2M\sin(\pi\xi/2)$, which exists in the whole stability region of the Coulomb gas, governs the large-distance asymptotic behaviour of the two-point correlation in (6). In particular, one has for the correlation function $h_{qq'}$ [20], as $r \to \infty$,

$$h_{qq'}(r) \sim qq' h(r) \qquad h(r) = -\lambda \left(\frac{\pi}{2m_1 r} \right)^{1/2} \exp(-m_1 r) \tag{19}$$

where λ is a β-dependent prefactor. The specific dependence of $h_{qq'}(r)$ on the charge product qq' means that the two-particle correlations are determined at large distance by the charge–charge correlation function $h_\rho(r) = \frac{1}{4} \sum_{q,q'=\pm} qq' h_{qq'}(r) = h(r)$.

On the other hand, the density correlation function $h_n = \frac{1}{4}\sum_{q,q'=\pm} h_{qq'}$ vanishes for the lowest one-B_1-breather state, and becomes nonzero only for the next two-B_1-breather states [21]. The mass of two B_1-breathers, $2m_1$, is smaller than that of the soliton–antisoliton pair, $2M$, in the region $0 < \beta < 1$. Consequently, as $r \to \infty$,

$$h_n(r) \propto \begin{cases} \exp(-2m_1 r) & \text{for} \quad 0 < \beta < 1 \\ \exp(-2Mr) & \text{for} \quad 1 \leqslant \beta < 2. \end{cases} \tag{20}$$

The correlation length depends continuously on β, but its first derivative with respect to β is discontinuous at $\beta = 1$. The large-distance exponential decay of h_n is faster than that of h_ρ. The two correlation lengths coincide just at the collapse point $\beta = 2$, where h_ρ and h_n differ from one another only by the inverse power law prefactors.

6. Miscellaneous topics and perspectives

The 2D Coulomb gas exhibits, at any temperature in the stability region, a universal finite-size correction to the grand potential as if we had a critical theory with the conformal anomaly number $c = -1$, although the particle correlation functions presented here decay exponentially. This phenomenon follows intuitively from the sine-Gordon representation of Ξ (5a), with the critical massless Gaussian field theory ($c = 1$) in the denominator. Explicit checks of the critical-like behaviour were done at the free-fermion $\beta = 2$ point for various geometries of confining domains [22–24], and at any $\beta < 2$ for the sphere [25, 26] and for the disc [21].

The ultimate task is to solve exactly the 2D Coulomb gas with a short-distance (maybe temperature-dependent) regularization of the pure Coulomb potential. We have made a first step towards this aim by deriving the leading correction to the exact bulk thermodynamics of pointlike charges due to the presence of the hard core of diameter σ around particles [27]. The results, which are conjectured to be exact in the low-density limit up to $\beta = 3$, reproduce correctly the σ-singularities of thermodynamic quantities at $\beta = 2$. They also confirm a 'subtraction' mechanism of singularities between the collapse point $\beta = 2$ and the KT transition point $\beta_{\mathrm{KT}} = 4$ within an ansatz proposed by Fisher *et al* [28] (excluding the existence of an infinite number of intermediate phases proposed in [29]), however, predict a different analytic structure of this ansatz. There exist candidates among integrable 1D quantum systems, for example, the lattice sine-Gordon model [30], which, after being formulated as a 2D Euclidean theory, might represent a Coulomb gas regularized in the whole temperature range.

Another topic, which has attracted much attention in the last few years due to the phenomenon of charge inversion [31], is charge asymmetry. The Coulomb gas with $|q_+| = 2|q_-|$ was solved in terms of the equivalent complex Bullough–Dodd field theory in [32]. Here, the fundamental changes in statistics caused by the charge asymmetry (e.g., the same correlation length for both correlation functions h_ρ and h_n at any stable β) were documented. This result might be a motivation for the exact solution of the 2D one-component plasma, which is the extreme charge-asymmetry case of the 2D Coulomb gas.

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